1.)

a.

Question: Construct from first principles the hamiltonian fore a 1D harmonic oscillator of mass m and spring constant k.

The kinetic energy is given by:

$$T = \frac{1}{2}m\dot{q}^2$$

While the potential energy is:

$$U = \frac{1}{2}kq^2$$

As we know. $L = T - U$ and $H = p\dot{q} - L$, leading to:

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{1}$$

or

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}m\omega q^2}$$

b.

Question: Determine the constant C such that $Q = C(p + im\omega q)$ and $P = C(p - im\omega q)$ define a canonical transformation.

There are multiple ways to show that a transformation is canonical, here I use the fact that Poisson brackets are conserved on canonical transforms.

$$[p, q] = [P, Q]$$

$$[p, q] = \frac{\partial p}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} = -1$$

$$[P, Q] = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} = (-im\omega C)C - C(im\omega C) = -2im\omega C^2$$

So we have:

$$-2im\omega C^2 = -1$$

$$\boxed{C = \sqrt{\frac{1}{2im\omega}}}$$

c.

Question: What is the generating function $S(q, P)$ for this transformation?

We have, by definition:

$$p = \frac{\partial S(q, P, t)}{\partial q} \tag{2}$$

$$Q = \frac{\partial S(q, P, t)}{\partial P} \tag{3}$$

So we can write:

$$p = im\omega q + \frac{P}{C} = \frac{\partial S}{\partial q}$$

$$S = \int \left(im\omega q + \frac{P}{C} \right) dq = \frac{Pq}{C} + \frac{im\omega}{2} q^2 + g(P) \tag{4}$$

Where $g(P)$ is some function depending only on P. Taking the derivative of Eq 4 with respect to P

$$\frac{\partial S}{\partial P} = \frac{q}{c} + \frac{dg}{dP}$$

And we know that the above should be equal to Q by Eq 3. So:

$$\frac{q}{c} + \frac{dg}{dP} = C(2im\omega q + \frac{P}{C})$$

$$\frac{dg}{dP} = \frac{q}{C} - \frac{q}{C} + P$$

$$g(P) = \frac{1}{2} P^2 \tag{5}$$

Finally, putting it all together:

$$\boxed{S(q, P) = \frac{1}{2} P^2 + \frac{qP}{C} + \frac{q^2}{4C^2}}$$

d.

Question: Find Hamilton's equations of motion for the new variables

We know that our new Hamiltonian, $\tilde{H}(Q, P, t)$, if related to our coordinates P and Q by:

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \tag{6}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} \tag{7}$$

But we can write $\dot{Q}$ and $\dot{P}$ as:

$$\dot{Q} = C(\dot{p} + im\omega\dot{q}) \tag{8}$$
$$\dot{P} = C(\dot{p} - im\omega\dot{q}) \tag{9}$$

and using our hamiltonian, $H(q,p,t)$ to derive $\dot{q}$ and $\dot{p}$:

$$\dot{Q} = C(-m\omega^2 q + im\omega\frac{p}{m}) = Ci\omega(p + im\omega q) = i\omega Q$$

Thus we can integrate the above equation to arrive at:

$$\tilde{H}(Q,P,t) = i\omega QP + g(Q) \tag{10}$$

where again $g(Q)$ is some function depending only on Q. Switching over to $\dot{P}$:

$$\dot{P} = C(\dot{p} - im\omega\dot{q}) = -i\omega P = -\frac{\partial\tilde{H}}{\partial Q}$$
$$-\frac{\partial\tilde{H}}{\partial Q} = -i\omega P + \frac{\partial g}{\partial Q}$$

Thus $g(Q) = 0$ and we can see that our new Hamiltonian is given by:

$$\boxed{\tilde{H}(Q,P,t) = i\omega QP}$$

Using this Hamiltonian, we can trivially see that:

$$\boxed{\dot{P} = -i\omega P}$$

$$\boxed{\dot{Q} = i\omega Q}$$

We can integrate these equations to find:

$$Q(t) = Q_0 \exp(i\omega t + \phi_1) \tag{11}$$
$$P(t) = P_0 \exp(-i\omega t + \phi_2) \tag{12}$$

or, substituting in for our original coordinates:

$$p(t) = \frac{1}{C}\left[Q_0 \exp(i\omega t + \phi_1) + P_0 \exp(-i\omega t + \phi_2)\right] \tag{13}$$
$$q(t) = \frac{1}{2im\omega C}\left[Q_0 \exp(i\omega t + \phi_1) - P_0 \exp(-i\omega t + \phi_2)\right] \tag{14}$$

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2. Fetter & Walecka 6.7

- canonical transformation $q_\sigma, p_\sigma \to Q_\sigma, P_\sigma$ preserves Hamilton's equations:

$$\Rightarrow \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma}, \quad -\dot{P}_\sigma = \frac{\partial \tilde{H}}{\partial Q_\sigma}$$

where $\tilde{H}$ is the new Hamiltonian in terms of the new generalized coordinates and momenta.

- require:

$$\delta \int_{t_1}^{t_2} dt \left\{ p_\sigma \dot{q}_\sigma - H(q,p,t) \right\} = 0 = \delta \int_{t_1}^{t_2} dt \left\{ P_\sigma \dot{Q}_\sigma - \tilde{H}(Q,P,t) \right\}$$

→ satisfied and phase-space volumes preserved when

$$(*) \qquad p_\sigma \dot{q}_\sigma - H(q,p,t) = P_\sigma \dot{Q}_\sigma - \tilde{H}(Q,P,t) + \frac{dF}{dt}$$

where F is the generating function

- for function $F_2(q,P,t)$ we have $F = F_2(q,P,t) - P_\sigma Q_\sigma$ (type 2 generating func.)

$$\Rightarrow \frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}$$

$$= \sum_\sigma \left(\frac{\partial F_2}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F_2}{\partial P_\sigma}\dot{P}_\sigma - Q_\sigma \dot{P}_\sigma - P_\sigma \dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

→ equation (*) can thus be written as

$$\sum_\sigma \left[\left(p_\sigma - \frac{\partial F_2}{\partial q_\sigma} \right) \dot{q}_\sigma + \left(Q_\sigma - \frac{\partial F_2}{\partial P_\sigma} \right) \dot{P}_\sigma \right] - H(q,P,t) = -\tilde{H}(Q,P,t) + \frac{\partial F_2}{\partial t}$$

⇒ need

$$p_\sigma - \frac{\partial F}{\partial q_\sigma} = 0, \quad Q_\sigma - \frac{\partial F_2}{\partial P_\sigma} = 0 \quad \Rightarrow \quad p_\sigma = \frac{\partial F_2}{\partial q_\sigma}, \quad Q_\sigma = \frac{\partial F_2}{\partial P_\sigma}$$

so that

$$\tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_2}{\partial t}$$

→

2. a) want to show

$$S_0(q,P) = \sum_\sigma q_\sigma P_\sigma$$

generates the identity transformation

- $S_0(q,P)$ is a type 2 generating function

$$\Rightarrow \begin{array}{l} p_\sigma = \dfrac{\partial F_2}{\partial q_\sigma} = P_\sigma \\ Q_\sigma = \dfrac{\partial F_2}{\partial P_\sigma} = q_\sigma \end{array} \Bigg\rangle \boxed{q_\sigma = Q_\sigma, \quad p_\sigma = P_\sigma} \checkmark$$

b) want to show the function $S_0 + Hdt$ generates the dynamical transformation from t to $t+dt$

- let $G \equiv S_0 + Hdt = \sum_\sigma q_\sigma P_\sigma + Hdt$

- again G is a type 2 generating function, so

Hamilton's Equations:

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad -\dot{p}_\sigma = \frac{\partial H}{\partial q_\sigma}$$

$$p_\sigma = \frac{\partial G}{\partial q_\sigma} = P_\sigma + \frac{\partial H}{\partial q_\sigma} dt$$

$$= P_\sigma - \dot{p}_\sigma dt \Rightarrow P_\sigma = p_\sigma + \dot{p}_\sigma dt$$

$$\boxed{P_\sigma = p_\sigma(t+dt)}$$

← new momenta are the old momenta evaluated at time $t+dt$ ✓

$$Q_\sigma = \frac{\partial G}{\partial P_\sigma} = q_\sigma + \frac{\partial H}{\partial P_\sigma} dt$$

$$\approx q_\sigma + \frac{\partial H}{\partial p_\sigma} dt \qquad \frac{\partial H}{\partial P_\sigma} \sim \frac{\partial H}{\partial p_\sigma} \text{ since } P_\sigma = p_\sigma(t+dt)$$

$$= q_\sigma + \dot{q}_\sigma dt$$

$$\boxed{Q_\sigma = q_\sigma(t+dt)}$$

← new coordinates are the old coordinates evaluated at time $t+dt$ ✓

⇒ Since we have shown that this generating function produces new coordinates and momenta satisfying $Q_\sigma = q_\sigma(t+dt)$ and $P_\sigma = p_\sigma(t+dt)$, the time development of any mechanical system must always be a canonical transformation. ✓

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HW # 3 Problem # 3

3) Fetter and Walecka 6.8

Let $G = S_0 + \underset{\sim}{P} \cdot d\underset{\sim}{r}$

$$G = \sum q_\sigma P_\sigma + \sum P_\sigma dq_\sigma$$

$$\frac{dG}{dt} = \sum (\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma) + \sum \dot{P}_\sigma dq_\sigma$$

Since this is a type II generating function we define:

$$F = G - \sum P_\sigma Q_\sigma$$

Canonical transformations satisfy the following equation:

$$\sum p_\sigma \dot{q}_\sigma - H = \sum P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{dF}{dt}$$

$$\sum p_\sigma \dot{q}_\sigma - H = \sum P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{dG}{dt} - \sum \dot{P}_\sigma Q_\sigma - \sum P_\sigma \dot{Q}_\sigma$$

$$\sum p_\sigma \dot{q}_\sigma - H = -\sum \dot{P}_\sigma Q_\sigma - \tilde{H} + \frac{dG}{dt}$$

$$\sum p_\sigma \dot{q}_\sigma - H = -\sum \dot{P}_\sigma Q_\sigma - \tilde{H} + \sum (\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma) + \sum \dot{P}_\sigma dq_\sigma$$

$$\sum \underbrace{(p_\sigma - P_\sigma)}_{=0 \quad (1)} \dot{q}_\sigma - H = \sum \underbrace{(-Q_\sigma + q_\sigma + dq_\sigma)}_{=0 \quad (2)} \dot{P}_\sigma - \tilde{H}$$

(1) $\Rightarrow$ $p_\sigma - P_\sigma = 0$

$$P_\sigma = p_\sigma$$

(2) $\Rightarrow$ $-Q_\sigma + q_\sigma + dq_\sigma = 0$

$$Q_\sigma = q_\sigma + dq_\sigma$$

$$\Rightarrow \left.\begin{array}{l} \underset{\sim}{P} = \underset{\sim}{p} \\ \underset{\sim}{Q} = \underset{\sim}{q} + d\underset{\sim}{r} \end{array}\right\} \text{Finite translations}$$

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3) Fetter and Walecka 6.8 continued

Let $G = S_0 + \hat{n}\cdot\underline{L}\,d\varphi \qquad \underline{L} = \underline{q}\times\underline{p}$

$$G = \sum q_\sigma P_\sigma + n_k \epsilon_{ijk}\, q_i P_j\, d\varphi$$

$$\frac{dG}{dt} = \sum(\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma) + n_k \epsilon_{\sigma jk}\,\dot{q}_\sigma P_j\, d\varphi + n_k \epsilon_{i\sigma k}\, q_i \dot{P}_\sigma\, d\varphi$$

Since this is a type II generating function we define:

$$F = G - \sum P_\sigma Q_\sigma$$

Canonical transformations satisfy the following equation:

$$\sum p_\sigma \dot{q}_\sigma - H = \sum P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{dF}{dt}$$

$$\sum p_\sigma \dot{q}_\sigma - H = \sum P_\sigma \dot{Q}_\sigma - \tilde{H} + \frac{dG}{dt} - \sum \dot{P}_\sigma Q_\sigma - \sum P_\sigma \dot{Q}_\sigma$$

$$\sum p_\sigma \dot{q}_\sigma - H = -\sum \dot{P}_\sigma Q_\sigma - \tilde{H} + \frac{dG}{dt}$$

$$\sum p_\sigma \dot{q}_\sigma - H = -\sum \dot{P}_\sigma Q_\sigma - \tilde{H} + \sum(\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma) + n_k \epsilon_{\sigma jk}\,\dot{q}_\sigma P_j\, d\varphi + n_k \epsilon_{i\sigma k}\, q_i \dot{P}_\sigma\, d\varphi$$

$$\underbrace{\sum(p_\sigma - P_\sigma - n_k \epsilon_{\sigma jk} P_j\, d\varphi)\dot{q}_\sigma}_{=0 \quad (1)} - H = \underbrace{\sum(-Q_\sigma + q_\sigma + n_k \epsilon_{i\sigma k}\, q_i\, d\varphi)\dot{P}_\sigma}_{=0 \quad (2)} - \tilde{H}$$

(2) $\Rightarrow$ $-Q_\sigma + q_\sigma + n_k \epsilon_{i\sigma k}\, q_i\, d\varphi = 0$

$$Q_\sigma = q_\sigma + n_k \epsilon_{i\sigma k}\, q_i\, d\varphi$$

$$Q_\sigma = q_\sigma + \epsilon_{ki\sigma}\, n_k q_i\, d\varphi$$

$$Q_\sigma = q_\sigma + [\hat{n}\times \underline{q}]_\sigma\, d\varphi$$

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3) Fetter and Walecka 6.8 continued

$$(1) \Rightarrow \quad p_\sigma - P_\sigma - n_k \epsilon_{\sigma j k} P_j \, d\varphi = 0$$
$$P_\sigma = p_\sigma + \epsilon_{\sigma j k} n_k P_j \, d\varphi$$
$$P_\sigma = p_\sigma + \epsilon_{k j \sigma} n_k P_j \, d\varphi$$
$$P_\sigma = p_\sigma + [\hat{n} \times P]_\sigma \, d\varphi$$

This equation can be written as $P_\sigma = p_\sigma + O(d\varphi)$ which indicates that replacing P_σ with p_σ introduces error on the order of $O(d\varphi)$

$$\Rightarrow P_\sigma = p_\sigma + [\hat{n} \times p]_\sigma \, d\varphi + O(d\varphi^2)$$

We ignore the $O(d\varphi^2)$ terms

$$P_\sigma = p_\sigma + [\hat{n} \times p]_\sigma \, d\varphi$$

$$\Rightarrow \left.\begin{aligned} Q &= q + [\hat{n} \times q] \, d\varphi \\ P &= p + [\hat{n} \times p] \, d\varphi \end{aligned}\right\} \text{Finite Rotations}$$

Fetter and Walecka 6.17

a) $G = \sum_r q_r P_r + \epsilon G(q_r, P_r, t)$

$$\frac{dG}{dt} = \sum_r (\dot{q}_r P_r + q_\sigma \dot{P}_\sigma) + \epsilon \frac{\partial G}{\partial q_r}\dot{q}_r + \epsilon \frac{\partial G}{\partial P_\sigma}\dot{P}_r + \epsilon \frac{\partial G}{\partial t}$$

Since this is type II transformation let $F = G - \sum P_\sigma Q_\sigma$

Canonical transformation must satisfy:

$$\sum p_r \dot{q}_\sigma - H = \sum P_r \dot{Q}_\sigma - \bar{H} + \frac{dF}{dt}$$

$$\sum p_\sigma \dot{q}_r - H = \sum P_\sigma \dot{Q}_\sigma - \bar{H} + \frac{dG}{dt} - \sum \dot{P}_\sigma Q_\sigma - \sum P_r \dot{Q}_\sigma$$

$$\sum p_\sigma \dot{q}_\sigma - H = -\sum \dot{P}_\sigma Q_\sigma - \bar{H} + \sum(\dot{q}_\sigma P_\sigma + q_\sigma \dot{P}_\sigma) + \epsilon\frac{\partial G}{\partial q_\sigma}\dot{q}_\sigma + \epsilon \frac{\partial G}{\partial P_r}\dot{P}_\sigma + \epsilon\frac{\partial G}{\partial t}$$

$$\sum \underbrace{\left(p_\sigma - P_\sigma - \epsilon\frac{\partial G}{\partial q_\sigma}\right)}_{0 \quad (1)} \dot{q}_\sigma - H = \sum \underbrace{\left(-Q_\sigma + q_\sigma + \epsilon\frac{\partial G}{\partial P_r}\right)}_{0 \quad (2)} \dot{P}_\sigma - \bar{H} + \epsilon\frac{\partial G}{\partial t}$$

(1) $\Rightarrow$ $p_\sigma - P_\sigma - \epsilon\frac{\partial G}{\partial q_\sigma} = 0$

$$P_r = p_\sigma - \epsilon \frac{\partial}{\partial q_\sigma} G(q_\sigma, P_\sigma, t)$$

Replacing P_σ with p_σ in G introduces error of $O(\epsilon^2)$

$$P_r = p_\sigma - \epsilon\frac{\partial}{\partial q_\sigma} G(q_r, p_r, t) + O(\epsilon^2)$$

(2) $\Rightarrow$ $Q_\sigma = q_\sigma + \epsilon \frac{\partial}{\partial P_\sigma} G(q_r, P_\sigma, t)$

Replacing P_σ with p_σ in G introduces error of $O(\epsilon^2)$

$$Q_\sigma = q_\sigma + \epsilon \frac{\partial p_\sigma}{\partial P_\sigma}\frac{\partial}{\partial p_\sigma} G(q_\sigma, p_\sigma, t) + O(\epsilon^2)$$

$$Q_\sigma = q_\sigma + \epsilon\left(1 + O(\epsilon)\right)\frac{\partial}{\partial p_\sigma} G(q_\sigma, p_\sigma, t) + O(\epsilon^2)$$

$$Q_\sigma = q_\sigma + \epsilon\frac{\partial}{\partial p_\sigma} G(q_\sigma, p_r, t) + O(\epsilon^2)$$

b) $F \rightarrow F + dF$

$$dF = \frac{\partial F}{\partial q_\sigma} dq_\sigma + \frac{\partial F}{\partial p_\sigma} dp_\sigma$$

From the cannonical transformation we have, to first order in ϵ:

$$dq_\sigma = \epsilon \frac{\partial G}{\partial p_\sigma} \quad , \quad dp_\sigma = -\epsilon \frac{\partial G}{\partial q_\sigma}$$

$$dF = \epsilon \frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \epsilon \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma}$$

$$dF = \epsilon [F, G]_{PB}$$

c) Under this transformations we have

$$H(q_\sigma, p_\sigma) \rightarrow H(q_\sigma, p_\sigma) + dH$$

$$H(q_\sigma, p_\sigma) \rightarrow H(q_\sigma, p_\sigma) + \epsilon [H, G]_{PB}$$

If G is a constant of motion then $\frac{\partial G}{\partial q_\sigma} = \frac{\partial G}{\partial p_\sigma} = 0 \Rightarrow [H, G]_{PB} = 0$

$$H(q_\sigma, p_\sigma) \rightarrow H(q_\sigma, p_\sigma)$$

Therefore the Hamiltonian is invariant under this transformation.

5. Soln:

a) We chose the third generating function

$$q = -\frac{\partial F_3(p,Q)}{\partial p} \qquad P = -\frac{\partial F_3}{\partial Q} = g(p)$$

$$\therefore \quad F_3 = -g(p)Q$$

$$q = -\frac{\partial F_3}{\partial p} = g'(p)Q$$

generating function: $F_3 = -g(p)Q$ ~~third~~ type 3 generator

transformation rule: $P = g(p)$

$$Q = q/g'(p)$$

To prove the phase space volume is invariant, we only need to verify the determinant of Jacobi Matrix is 1

i.e. $$\det\left|\frac{\partial \varepsilon_i}{\partial \Lambda_j}\right| = 1$$

$$\frac{\partial Q}{\partial q} = \frac{1}{g'(p)} \qquad \frac{\partial Q}{\partial p} = -\frac{q g''(p)}{(g'(p))^2} \qquad \frac{\partial P}{\partial q} = 0 \qquad \frac{\partial P}{\partial p} = g'(p)$$

$$\therefore \quad \det\left|\frac{\partial \varepsilon_i}{\partial \Lambda_j}\right| = \frac{1}{g'(p)}\cdot g'(p) - 0 = 1 \qquad \text{Q.E.D}$$

b) Under canonical transformation, principle of least action should still holds.

i.e. $$\delta\int_{t_1}^{t_2}(p\dot{q} - H) = \delta\int_{t_1}^{t_2} dt\,(P\dot{Q} - \breve{H})$$

which means $$p\dot{q} - H = P\dot{Q} - \breve{H} + \frac{dF}{dt}$$

$$\therefore \quad \breve{H} = H + P\dot{Q} - p\dot{q} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}$$

where $$-P = \frac{\partial F}{\partial Q} \qquad +p = \frac{\partial F}{\partial q} \qquad \frac{\partial F}{\partial p} = \frac{\partial F}{\partial P} = 0$$

then $$\breve{H} = H + \frac{\partial F}{\partial t}$$

$$\therefore \quad \frac{-\partial \breve{H}}{\partial Q} = -\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial Q}\right) = +\dot{P} \qquad \text{Q.E.D}$$

6. $H=\frac{1}{2m}(p_1^2+p_2^2+p_3^2)+\frac{1}{2}k_1q_1^2+\frac{1}{2}k_2q_2^2+\frac{1}{2}k_3q_3^2=E$

Seperate the eqn.

$$\begin{cases}\frac{1}{2m}p_1^2+\frac{1}{2}k_1q_1^2=E_1\\ \frac{1}{2m}p_2^2+\frac{1}{2}k_2q_2^2=E_2\\ \frac{1}{2m}p_3^2+\frac{1}{2}k_3q_3^2=E_3\end{cases} \qquad \sum_{i=1}^{3}E_i=E \quad . \quad p_i=\sqrt{2m}\,(E_i-\tfrac{1}{2}k_iq_i^2)^{1/2}$$

$$I_i=\frac{1}{2\pi}\oint p_i\,dq_i$$

$$=\frac{\sqrt{2m}}{2\pi}\oint(E_i-\tfrac{1}{2}k_iq_i^2)^{1/2}dq_i$$

$$q_i=\left(\frac{2E_i}{k_i}\right)^{1/2}\sin\theta \quad , dq_i=\left(\frac{2E_i}{k_i}\right)^{1/2}\cos\theta\,d\theta$$

$$\therefore I_i=\frac{E_i}{\pi}\sqrt{\frac{m}{k_i}}\int\cos^2\theta\,d\theta \quad =\frac{E_i}{\sqrt{k_i/m}}$$

$$\therefore E=\sum_{i=1}^{3}E_i=I_1\sqrt{\frac{k_1}{m}}+I_2\sqrt{\frac{k_2}{m}}+I_3\sqrt{\frac{k_3}{m}} \quad \Rightarrow \begin{cases}w_1=\frac{\partial H}{\partial I_1}=\frac{\partial E}{\partial I_1}=\sqrt{\frac{k_1}{m}}\\ w_2=\sqrt{\frac{k_2}{m}}\\ w_3=\sqrt{\frac{k_3}{m}}\end{cases}$$

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PHYS 200B

7. The reduced form of the variational principle (by Maupertuis) uses the abbreviated action:

$$S_0 = \int p\,dq \qquad \text{or} \qquad I = \frac{1}{2\pi}\int p\,dq$$

We found that I is an adiabatic invariant, that is, $\frac{dI}{dt} = 0$ for fixed E and λ, where $\lambda(t)$ is a slowly varying parameter such that:

$$\frac{\dot{\lambda}}{\lambda} << \frac{1}{T} \quad \text{and } T \text{ is a fast time scale}$$

For the harmonic oscillator we can have the frequency, $\omega(t)$, slowly varying:

$$L = \frac{1}{2}m\dot{x}^2 - m\omega^2(t)\,x^2, \quad H = \frac{p^2}{2m} + \frac{m}{2}\omega^2(t)\,x^2$$

Note we still have $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$. Over the fast time scale $T = \frac{2\pi}{\omega}$:

$$I = \frac{1}{2\pi}\oint p\,dx = \frac{1}{2\pi}\int_t^{t+T}(m\dot{x})(\dot{x}\,dt')$$

At fixed $\omega(t)$, assuming x (thus $\dot{x}$) periodic in ω:

$$\frac{dI}{dt} = \frac{m}{2\pi}\int_t^{t+T}\frac{d}{dt'}(\dot{x})^2\,dt' = \frac{m}{2\pi}\,\dot{x}^2\Big|_t^{t+T} = 0$$

At fixed ω, $x = a\cos(\omega t + \varphi)$ and:

$$I = \frac{m}{2\pi}\int_t^{t+T}\omega^2 a^2 \sin^2(\omega t + \varphi)\,dt' = \frac{m\omega^2}{2\pi}\,\frac{\pi}{\omega} = \frac{m\omega a^2}{2}$$

but $E = \frac{1}{2}m\omega^2 a^2$, so also $I = \frac{E}{\omega}$. So:

$$I\omega(t) = E(t) = \frac{1}{2}m\omega^2 a^2 \quad \text{thus} \quad a = \frac{2I}{m\sqrt{\omega}} = \frac{c}{\sqrt{\omega}}$$

where c is a constant.

Thus, a reasonable solution to the problem of varying $\omega(t)$ should have the amplitude as $\omega^{-1/2}$. Indeed, we can apply the WKB method, since

$$\left|\frac{\dot{\omega}}{\omega}\right| = \left|\dot{T}/T\right| = \left|\overbrace{\frac{dT}{dt}}^{\sim\varepsilon} \cdot \frac{\omega}{2\pi}\right| \sim O(\varepsilon\omega) << \frac{1}{T} \quad \text{with } \varepsilon \text{ a small parameter}$$

So the solution is the same except we replace the space variable, x, with time, t:

$$x^{\pm}(t) = \frac{C_\pm}{\sqrt{\omega(t)}} e^{\pm i\int \omega(t)dt} \quad \text{and generally}$$

$$x = \frac{1}{\sqrt{\omega(t)}}\left(C_+ e^{+i\int\omega(t)dt} + C_- e^{-i\int\omega(t)dt}\right) = \frac{C}{\sqrt{\omega(t)}}\cos\left(\int\omega(t)\,dt + \varphi\right)$$

Either form that is used the amplitude has a factor of $\omega(t)^{-1/2}$, so:

$\bar{I}$

$$\frac{d\bar{I}}{dt} = \frac{d}{dt}\left(\frac{\bar{E}}{\bar{\omega}}\right) = \frac{d}{dt}\left(\frac{\frac{1}{2}m\bar{\omega}^2 a^2}{\bar{\omega}}\right)$$

$$= \frac{d}{dt}\left(\frac{\frac{1}{2}m\bar{\omega}^2(C/\omega)}{\bar{\omega}}\right) = \frac{d}{dt}\left(\frac{1}{2}mC\frac{\omega^2}{\omega^2}\right)$$

$$= \boxed{0} \quad \left(\bar{E}, \bar{I}, \bar{\omega} \text{ denotes average from } t \text{ to } (t+T) \text{ and } \omega(t) \simeq \bar{\omega}\right)$$

Above C, $C\pm$, were constants determined by initial conditions.

More directly:

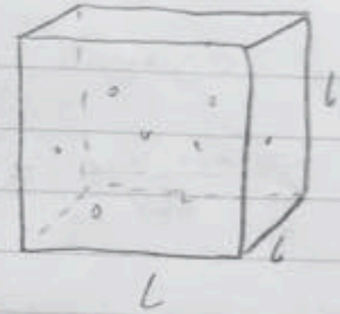
$$I = \frac{1}{2\pi}\oint_t p\,dq = \frac{m}{2\pi}\int_t^{t+T}\dot{x}^2dt' \approx \frac{m}{2\pi}\int_t^{t+T}\Big[\overbrace{\frac{\dot{\omega}}{\omega^{3/2}}x}^{0} - \frac{C}{\omega^{1/2}}\cdot\omega\sin\Big(\int_0^{t'}\omega\,dt'' + \varphi\Big)\Big]^2 dt'$$

Use that ω is slowly varying:

$$I \approx \frac{mC^2}{2\pi}\left(\frac{\omega}{\omega^{1/2}}\right)^2 \cdot \int_t^{t+T}\sin^2(\bar{\omega}t' + \varphi)\,dt' = \frac{mC^2}{2\pi}(\omega)\cdot\frac{\pi}{\omega} = \frac{mC^2}{2} \quad (\text{a constant})$$

Prob 8.

Point particles in box with sides L!



Our goal is to use adiabatic theory to find the relation between pressure and volume when the walls are moving.

Start with the stationary case:

Assume particles are either moving in x, y or z direction, and that its the same number of particles in each direction.

To find pressure we use $P = \frac{F}{A}$

The force is given by $F = \frac{\Delta p}{\Delta t}$.

Δp is the change in momentum due to the collision, while Δt is the time between each collision.

Energ of particles in x-dirextion:

$$E_x = \sum_k \frac{1}{2} m_k v_{kx}^2$$

Change in moment due to collision:

$$\Delta p = \sum_k 2 m_k v_{kx}$$

Time between collision for one particle:

Force on wall is then given by:

$$F_x = \sum_k \frac{2m_k v_{kx}}{2L/v_{kx}} = \frac{2}{L}\sum_k \frac{1}{2}m_k v_{kx}^2 = \frac{2E_x}{L}$$

Since we asumed $\frac{1}{3}$ of the particles to move in x direction $E_x = \frac{1}{3}E$

Whe can then find the pressure:

$$P = \frac{F_x}{A} = \frac{2E_x}{L^3} = \underline{\frac{2}{3}\frac{E}{V}}$$

Now lets take the moving walls into account:

We asume that the rate of change of the walls are much slower tan the rate of collision. This means we kan use adiabatic theory!

Adiabatic invariant: $I = \oint \frac{p\,dq}{2\pi}$

We look at motion in the x direction. Since this is free particles the countur integral reduces to a line integral in two directions:

$$2\pi \cdot I = \int_0^L p_x dx + \int_L^0 (-p_x)\,dx = 2p_x L = 2L\sqrt{2mE_x} = \text{const}$$

We can then simplify this to $E_x L^2 = \text{const}$

Taking the differential yields:

$$dE_x L^2 + 2LE_x dL = 0 \Rightarrow dE_x = -\frac{2E_x}{L}dL$$

Since the variation is slow we can asume that $P = \frac{2}{3}\frac{E}{V} = \frac{2E_x}{V}$ still holds

This allow us to write

$$dE_x = -\frac{2PV}{2L}dL$$

We have $V = L_x L_y L_z$, $dV = L_y L_z dL_x = L^2 dL$

$$\Rightarrow dE = -\frac{PV}{V}dV = -PdV$$

This resault is agrees with the first law of thermodynamics for an adiabatic expansion!

1.)

a.

Question: Construct from first principles the hamiltonian fore a 1D harmonic oscillator of mass m and spring constant k.

The kinetic energy is given by:

$$T = \frac{1}{2}m\dot{q}^2$$

While the potential energy is:

$$U = \frac{1}{2}kq^2$$

As we know. $L = T - U$ and $H = p\dot{q} - L$, leading to:

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{1}$$

or

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}m\omega q^2}$$

b.

Question: Determine the constant C such that $Q = C(p + im\omega q)$ and $P = C(p - im\omega q)$ define a canonical transformation.

There are multiple ways to show that a transformation is canonical, here I use the fact that Poisson brackets are conserved on canonical transforms.

$$[p, q] = [P, Q]$$
$$[p, q] = \frac{\partial p}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} = -1$$
$$[P, Q] = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} = (-im\omega C)C - C(im\omega C) = -2im\omega C^2$$

So we have:

$$-2im\omega C^2 = -1$$

$$\boxed{C = \sqrt{\frac{1}{2im\omega}}}$$

c.

Question: What is the generating function $S(q,P)$ for this transformation?

We have, by definition:

$$p = \frac{\partial S(q,P,t)}{\partial q} \tag{2}$$

$$Q = \frac{\partial S(q,P,t)}{\partial P} \tag{3}$$

So we can write:

$$p = im\omega q + \frac{P}{C} = \frac{\partial S}{\partial q}$$

$$S = \int \left(im\omega q + \frac{P}{C} \right) dq = \frac{Pq}{C} + \frac{im\omega}{2} q^2 + g(P) \tag{4}$$

Where $g(P)$ is some function depending only on P. Taking the derivative of Eq 4 with respect to P

$$\frac{\partial S}{\partial P} = \frac{q}{c} + \frac{dg}{dP}$$

And we know that the above should be equal to Q by Eq 3. So:

$$\frac{q}{c} + \frac{dg}{dP} = C(2im\omega q + \frac{P}{C})$$

$$\frac{dg}{dP} = \frac{q}{C} - \frac{q}{C} + P$$

$$g(P) = \frac{1}{2}P^2 \tag{5}$$

Finally, putting it all together:

$$\boxed{S(q,P) = \frac{1}{2}P^2 + \frac{qP}{C} + \frac{q^2}{4C^2}}$$

d.

Question: Find Hamilton's equations of motion for the new variables

We know that our new Hamiltonian, $\tilde{H}(Q,P,t)$, if related to our coordinates P and Q by:

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \tag{6}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} \tag{7}$$

But we can write $\dot{Q}$ and $\dot{P}$ as:

$$\dot{Q} = C(\dot{p} + im\omega\dot{q}) \tag{8}$$
$$\dot{P} = C(\dot{p} - im\omega\dot{q}) \tag{9}$$

and using our hamiltonian, $H(q,p,t)$ to derive $\dot{q}$ and $\dot{p}$:

$$\dot{Q} = C(-m\omega^2 q + im\omega\frac{p}{m}) = Ci\omega(p + im\omega q) = i\omega Q$$

Thus we can integrate the above equation to arrive at:

$$\tilde{H}(Q,P,t) = i\omega QP + g(Q) \tag{10}$$

where again $g(Q)$ is some function depending only on Q. Switching over to $\dot{P}$:

$$\dot{P} = C(\dot{p} - im\omega\dot{q}) = -i\omega P = -\frac{\partial\tilde{H}}{\partial Q}$$
$$-\frac{\partial\tilde{H}}{\partial Q} = -i\omega P + \frac{\partial g}{\partial Q}$$

Thus $g(Q) = 0$ and we can see that our new Hamiltonian is given by:

$$\boxed{\tilde{H}(Q,P,t) = i\omega QP}$$

Using this Hamiltonian, we can trivially see that:

$$\boxed{\dot{P} = -i\omega P}$$

$$\boxed{\dot{Q} = i\omega Q}$$

We can integrate these equations to find:

$$Q(t) = Q_0 \exp(i\omega t + \phi_1) \tag{11}$$
$$P(t) = P_0 \exp(-i\omega t + \phi_2) \tag{12}$$

or, substituting in for our original coordinates:

$$p(t) = \frac{1}{C}\left[Q_0 \exp(i\omega t + \phi_1) + P_0 \exp(-i\omega t + \phi_2)\right] \tag{13}$$
$$q(t) = \frac{1}{2im\omega C}\left[Q_0 \exp(i\omega t + \phi_1) - P_0 \exp(-i\omega t + \phi_2)\right] \tag{14}$$

4.)

a.) Pondermotive Force:

$\frac{\dot{\lambda}}{\lambda} >> \Omega$ where Ω is natural frequency

We make the assumption that the motion, $x(t)$, can be separated into a fast plus slow component, $\bar{x} + \tilde{x}$. Then the key part comes by averaging over the fast period. This allows us to find mean-field equations and a form for the effective potential.

Key Features:

New positions of stability/instability, range of driving frequencies for which these positions of equilibrium exist.

The canonical example:

Pendulum with a driven (oscillating) support. Inverted pendulum

Summary of result:

$$U_{eff} = U + \frac{1}{4m\omega^2}\left(f_1^2(y) + f_2^2(y)\right)$$

$$\frac{dU_{eff}}{dy} = 0 \quad \& \quad \frac{d^2U_{eff}}{dy^2} > 0 \quad \text{for stabilization}$$

b.) Parametric Instability:

$\frac{\dot{\lambda}}{\lambda} \simeq 2\Omega$

We assume we can treat the solution as fast oscillator term modulated by a slowly growing amplitude. I.e. $x(t) = a(t)\sin(\omega t) + b(t)\cos(\omega t)$ where $a(t), b(t)$ vary slowly with respect to the frequency ω. Also driving near resonance.

We essentially treat the solution as a small perturbation from the standard, non-forcing solution. If the driving amplitude is small compared to the dimensions of the system, this perturbation is allowed.

Key Features:

Growth equation of the slowly varying amplitudes allow for both stable and divergent growth. $\underbrace{\omega_0^2 h^2}_{\text{Amplitude of parametric freq}}$ vs $\underbrace{\epsilon^2}_{\text{Mismatch}}$ So sufficiently close to resonance leads to growth

9.) continued

There is a range, or rather a threshold, for instability.

Example:

Driven oscillator with sinusoidally varying frequency. $\ddot{x} + \omega_0^2 [1 + h\cos((2\omega_0 + \epsilon)t)] x = 0$

Summary of example:

Growth parameter $S^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4}\left(\frac{\omega_0^2 h^2}{4} - \epsilon^2\right)$

For $\epsilon^2 >> \frac{\omega_0^2 h^2}{4}$, stable oscillation.

c.) Adiabatic Invariance;

$$\frac{\dot{\lambda}}{\lambda} << \Omega$$

We assume that the variation of λ is small over the natural period. Compared to the natural frequency, the parameter λ is taken to be const.

The leverage here is time scale separation, we can break the average

$$\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{\partial \lambda}{\partial t}} = \overline{\frac{\partial H}{\partial \lambda}}\frac{\partial \lambda}{\partial t} \quad \text{or} \quad \frac{d\bar{E}}{dt} = \frac{\partial \lambda}{\partial t}\left\langle \frac{\partial H}{\partial \lambda} \right\rangle_{cycle}$$

By taking $\lambda \approx$ const over one period, we ensure "phase" symmetry $\Rightarrow$ conserved charge.

Averaging over the period of the motion, smooths the variations in E.

Key Features:

For a fixed energy, and for slowly varying λ, there is an adiabatic invariant

$$I = \oint_{E,\lambda} \frac{1}{2\pi} p\, dq = \text{const} \qquad \text{Conservation of phase space area!}$$

Example:

Slowly varying natural frequency of an oscillator.

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$$

Line Summary:

$$I = \oint_{E,\omega} \frac{1}{2\pi} p\, dq = \text{const} = \frac{1}{2\pi} \pi \left(\sqrt{\frac{2E}{m\omega^2}}\right)\left(\sqrt{2mE}\right) = \frac{E}{\omega}$$

$$I = \frac{E}{\omega} = \text{const} \quad \text{so Energy is proportional to frequency}$$

9.) continued

d.) Anharmonic Oscillator

Expansion parameter $\epsilon << \omega_0^2$

We assume we can write the solutions $x(t)$ and ω as a series of successive approximations

$x(t) = x^{(0)} + x^{(1)} + x^{(2)} + \dots$ where $x^{(i)}$ is $\mathcal{O}(\epsilon^i)$

$\omega = \omega_0 + \omega^{(1)} + \omega^{(2)}$ $\qquad \omega^{(i)}$ is $\mathcal{O}(\epsilon^i)$

The main issue here is the we have a beat phenomenon from the x^3 term, thus we can have resonance and thus a divergence. The key is to use the expansion and choose $\omega^{(1)}, \omega^{(2)}$, etc so the resonant terms disappear. [Reductive perturbation]

Key Feature:

Non-linear frequency shift!

Example:

Duffing eqn: $\ddot{x} + \omega_0^2 x + \beta x^3 = 0$

Summary:

A non-linear freq shift occurs, $\omega = \omega_0 + \underbrace{\frac{3 a^2 \beta}{8 \omega_0}}_{\to \, \propto a^2 !!}$

Table Summary:

Case:	Time Scales	Approx Made	The Leverage	Key Features
Pondermotive Force	$\frac{\dot{\lambda}}{\lambda} >> \Omega$	$x(t) = \bar{x}(t) + \tilde{x}(t)$ $U \simeq U_{eff}$	Average over fast period to find eff. potential	New positions of stability/instability
Parametric Instability	$\frac{\dot{\lambda}}{\lambda} \simeq 2\Omega$	$x(t) = a(t)\cos(\omega t) + b(t)\sin(\omega t)$ $a(t), b(t)$ slowly varying	Treat the soln as a perturbation from unperturbed soln.	Growth of the amplitudes all for stable & divergent behav. $\underset{\text{Amp}}{\omega_0^2 h^2}$ vs $\underset{\text{mismatch}}{\epsilon^2}$
Adiabatic Invariance	$\frac{\dot{\lambda}}{\lambda} << \Omega$	$\lambda \simeq$ const over period T	Break up the average $\frac{\partial \bar{E}}{\partial \lambda} = \frac{\partial \bar{H}}{\partial \lambda} \frac{\partial \lambda}{\partial t}$	$I = \oint_{E,\lambda} \frac{1}{2\pi} p \, dq = \text{const}$ Conservation of phase space area
Anharmonic Oscillator	$\epsilon << \Omega^2$	$x = x^{(0)} + x^{(1)} + \dots$ $\omega = \omega^{(0)} + \omega^{(1)} + \dots$	Freedom of choosing $\omega^{(1)}, \omega^{(2)}$, etc. so resonant term disappears.	Non-linear frequency shift

9) continued

Case	Example	Line Summary
Pond. Force	Inverted pendulum with a driven support	$U_{eff} = U + \frac{1}{4m\omega^2}\left(f_1^2(y) + f_2^2(y)\right)$ $\frac{dU_{eff}}{dy} = 0$ & $\frac{d^2U_{eff}}{dy^2} > 0$ for stabilization
Parametric Instability	Driven Oscillator with Driving freq ≈ $2\omega_0$	Growth parameter: $S^2 = \frac{1}{4}\left(\frac{\omega_0^2h^2}{4} - \epsilon^2\right)$ Stable for $\epsilon^2 >> \frac{\omega_0^2h^2}{4}$
Adiabatic Invariance	Slowly varying ω_0 of oscillator	$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\sqrt{\frac{2mE}{1}}\sqrt{\frac{2E}{m\omega^2}} = E/\omega = \text{const}$
Anhar. Oscillator	Duffing eqn $\ddot{x} + \omega_0^2 x + \beta x^3 = 0$	$\omega = \omega_0 + \frac{3}{8}\frac{a^2 z}{\omega_0}$